ON THE CANONICAL FORMULA OF C. LÉVI-STRAUSS

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'No', was the answer. 'We have come to give you metaphors for poetry'. W.B. Yeats, **A Vision**

This note sketches a formal mathematical interpretation for the anthropologist Claude Lévi-Strauss's 'canonical formula'

$$F_x(a) : F_y(b) \sim F_x(b) : F_{a^{-1}}(y)$$

which he has found useful in analysing the structure of myths. Maranda's volume [10] is a useful introduction to this subject, which has a somewhat controversial history [5] among mathematicians. In view of the perspective proposed here (in terms of finite non-commutative groups), I believe that skepticism is quite understandable. Nevertheless, I believe that Lévi-Strauss knows what he means to say, and that difficulties in interpreting his formula are essentially those of translation between the languages of disciplines (anthropology and mathematics) that normally don't engage in much conversation.

I am posting this in the hope of encouraging such dialog. This document is addressed principally to mathematicians, and does not attempt to summarize any anthropological background; but in hopes of making it a little more accessible to people in that field, I have spelled out a few technical terms in more detail that mathematicians might think necessary.

I believe that Lévi-Strauss perceives the existence of a nontrivial anti-automorphism of the quaternion group of order eight; the latter is a mathematical object similar to, but more complicated than, the Klein group of order four which has appeared elsewhere in his work, cf. eg. [7 §6.2 p. 403], [14 p. 135]. In the next few paragraphs I will define some of these terms, and try to explain why I believe the formulation above captures what LS means to say. I want to thank the anthropologist Fred Damon for many discussions about this topic, and in particular for drawing my attention to Maranda's volume, which has been extemely helpful.

1 I was especially happy to find, in at least three separate places in that book, anthropologists raising the question of whether the two sides of the canonical formula are intended to be understood on a symmetrical footing [see [12 p. 35], [13 p. 83], and, most clearly, [3 p. 202]]; in other words, whether the symbol $\sim$ in Lévi-Strauss's formula is meant to be an equivalence relation in the mathematical sense.

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If formal terms, an equivalence relation is a relation between two objects (to be concrete: triangles in the plane), with the properties that

i) if object A is related to object B (in symbols: $A \sim B$), and object B is related to object C (ie $B \sim C$) then necessarily $A \sim C$, ie object A is related to object C;

ii) if A is related to B, then B is similarly related to A (this is the axiom of symmetry, which can be stated symbolically: if $A \sim B$ then $B \sim A$), and finally

iii) the axiom of reflexivity: any object is related to itself, ie $A \sim A$.

This notion permits us to disguish equivalence from identity; thus in plane geometry the symbol $\sim$ is traditionally used for the relation of **similarity**, which means that two triangles have the same angles, but are not necessarily of the same size. The point is that things can resemble each other in various ways [size, color,...], and that there may be good reason to compare differing **sorts** of equivalences (in the way that CLS says that his formula is about 'analogies between analogies'). What Côté, Racine, and Schwimmer all suggest is an interpretation of the canonical formula in which the right-hand side is a **transformation** of the left; in more standard mathematical notation, this might be written

$$F_x(a) : F_y(b) \to F_x(b) : F_{a^{-1}}(y)$$

The existence of such a transformation turning the left side into the right does not preclude that transformation from being an equivalence; all it does is allow us to regard axiom ii) above as optional. This fits quite naturally with current thinking about category theory.

2 To explain why this is relevant involves a short digression about the roles of the two **characters** a, b and the two **functions** x, y in the formula, cf. [13 p. 73-76]. This is in turn related to the role of symbolic notation in the first place.

Mathematicians use what they call **variables**, ie symbols such as a, b, x, y, to express relations which hold for a large class of objects, cf. [1 Ch. A §7]; but to interpret these relations, it is necessary to understand where the assertion is intended to hold. Thus, for example, a relation which makes sense for triangles might not make sense for real numbers. Another important background issue in the interpretation of mathematical formulas is the role of what are called **quantifiers**, which tell us whether (for example) the formula is intended to hold for **every** object in an appropriate class, or perhaps only that **some** object exists, for which the relation holds.

In the case of the canonical formula, this is particularly important, because such background information about quantifiers and domain of validity has been left unspecified. The formula requires one of its characters (a) to have an associated function (a^{-1}), cf. [12 p. 83], and it also requires one of the functions (y) to play the role of a character, on the right side of the formula: this is a key part of the formula's assertion, in some ways its central essential double twist. The formula is thus intrinsically **un**symmetric: it is not required that the character b have an associated function b^{-1}, nor that the function x have a sensible interpretation as a character. This suggests that the canonical formula can be paraphrased as the assertion:

• In a sufficiently large and coherent body of myths we can identify characters a, b and functions x, y, such that the mythical system defines a transformation which sends a to b, y to a^{-1}, and b to y, while leaving x invariant.

This transformation will therefore send the ratio, or formal analogy, $F_x(a) : F_y(b)$ into the ratio $F_x(b) : F_{a^{-1}}(y)$: this is the usual statement of the formula.

3 I don't think this is very controversial; several of the contributors to [10] have suggested similar interpretations. I have gone into the question in some detail, however, to make the point that **if** we can treat the right-hand side of the canonical formula on an equal footing with the left-hand side, we should then be able to apply the canonical formula **again**; but with b now as the new a, y as the new b, and a^{-1} as the new y, defining a chain

$$F_x(a) : F_y(b) \to F_x(b) : F_{a^{-1}}(y) \to F_x(y) : F_{b^{-1}}(a^{-1}) \ .$$

which is consistent with the interpretation of $F_x(a)$ as kind of ratio x/a of x to a: the left and right-hand sides of the chain above then become the valid rule

$$\frac{x/a}{y/b} = \frac{x/y}{b^{-1}/a^{-1}}$$

for the manipulation of grammar-school fractions. Mosko's [11] variant

$$F_x(a) : F_y(b) \sim F_x(b) : F_y(a) \ ,$$

of Lévi-Strauss's formula also has such an interpretation, when the algebraic values assigned to the variables lie in a commutative group in which every element has order two, neutralizing the opposition between a and a^{-1}. This presents Mosko's equation as a version of the CF valid in particularly symmetrical situations.

4 I've spelled this argument out because it suggests that interpreting the canonical formula as expressing the existence of a transformation relating its two sides is a useful idea. The remainder of this note will be concerned with the quaternion group mentioned in the first paragraph, as an example of a consistent classical mathematical system exemplifying Lévi-Strauss's formula.

It may be useful to say here that a **group**, in mathematical terminology, is a system of 'elements' (real numbers, for example), together with a system of rules for their combination (eg addition). There are lots of such critters, and some of them are **not** commutative, in the sense that the order in which we combine the elements may be significant. In the case of real numbers, order is not important (and hence it's conceivable one's checkbook might balance); but rotations in three-dimensional space (cf. Rubik's cube) form another example of a group, in which the order of operations **is** important.

The quaternion group of order eight (there are other quaternion groups, cf. [2 §5.2]) is the set

$$Q = \{\pm 1, \ \pm i, \ \pm j, \ \pm k\} \ ,$$

with a noncommutative law of multiplication, in which the product of the elements i and j (in that order) is k, but the product in the opposite order is $-k$; in other words,

$$i \cdot j = k = -j \cdot i, \ j \cdot k = i = -k \cdot j, \ k \cdot i = j = -i \cdot k \ .$$

To complete the 'multiplication table' for this group, we have to add the relations

$$i \cdot i = j \cdot j = k \cdot k = -1 \ ,$$

as well (last but not least) as the relation $(-1)^2 = +1$. The Klein group K , which has appeared previously in Lévi-Strauss's work, can be similarly described, as a **commutative** version of the group Q; in other words, the multiplication table is as before, except that we don't bother with the plus and minus signs:

$$K = \{1, i, j, k\} \ ,$$

given the simpler multiplication table

$$i \cdot j = k = j \cdot i, \ j \cdot k = i = k \cdot j, \ k \cdot i = j = i \cdot k \ ,$$

together with the relations

$$i \cdot i = j \cdot j = k \cdot k = 1$$

(and of course relations like $1 \cdot i = i = i \cdot 1$, etc.).

Two groups are **isomorphic** if their elements correspond in a way which preserves the multiplication laws: thus in Lévi-Strauss's writings the Klein group is described as the set of transformations which send a symbol x to the possible values $x, \ -x, \ 1/x, \ -1/x$; the first such transformation $[x \to x]$ corresponds to the 'identity' element 1 in the presentation of K given above, while the second, ie $x \to -x$, corresponds to the element i; similarly $x \to 1/x$ corresponds to j, etc. It is straightforward to check that the multiplication tables of these two structures correspond, eg the composition of the transformations $x \to -x$ (corresponding to i) with the composition $x \to 1/x$ (corresponding to j) is the transformation $x \to 1/(-x) = -1/x$ corresponding to k, and so forth.

Similarly, an **anti-**isomorphism of groups is an invertible transformation which reverses multiplication: it is a map which sends the product of any two elements g, h (in that order) to the product of the image elements, in the **reverse** order. In the case of commutative groups, this is a distinction without a difference, but in the case of a noncommutative group such as Q, it can be significant.

5 For example: the transformation $\lambda : Q \to Q$ which sends i to k, j to $i^{-1} = -i$, and k to j is a nontrivial example of an antiautomorphism: for example,

$$\lambda(i \cdot j) = \lambda(k) = j = (-i) \cdot k = \lambda(j) \cdot \lambda(i) \ ,$$

while

$$\lambda(j \cdot k) = \lambda(i) = k = j \cdot (-i) = \lambda(k) \cdot \lambda(j) \ ,$$

etcetera. Once this is established, it is easy to check that the assignment

$$x \mapsto 1, \ a \mapsto i, \ y \mapsto j, \ b \mapsto k$$

sends the antiautomorphism λ to the transformation

$$x \to x, \ a \to b, \ y \to a^{-1}, b \to y$$

defining the canonical formula.

6 Quod, as we say in the trade, erat demonstrandum: this presents an example of a consistent mathematical system, satisfying a version of Lévi-Strauss's formula.

It is a standard principle of mathematical logic, that the consistency of a system of axioms can be verified by giving just **one** example of an interpretation in which those axioms hold true; but I believe that in this case, there may be more to the story. Logicians are concerned with questions of logical truth, which can be formulated in terms of the commutative group $\{\pm 1\}$ [which can alternately be described in terms of two-valued 'yes-no' judgements, or in terms of the even-odd distinction among integers]. The Klein group is an interesting kind of 'double' of this group, with four elements rather than two, and the quaternion group takes this doubling process yet one step further. Something similar seems to occur in the study of kinship structures [15], but the groups encountered in that field remain necessarily commutative.

7 I believe the interpretation proposed here is also helpful in understanding another aspect of the canonical-formula problem, which other commentators have also found confusing: in [8 Ch. 6 p. 156], Lévi-Strauss invokes the formula

$$F_x(a) : F_y(b) \sim F_y(x) : F_{a^{-1}}(b) \ .$$

This differs from the previous version: now x on the left of the equation becomes y on the right, while a on the left becomes x on the right, y is transformed into a^{-1}, and finally b remains invariant. In the framework of paragraph six above, the assignment

$$x \mapsto i, \ y \mapsto j, \ a \mapsto k, \ b \mapsto 1$$

expresses this transformation as another anti-automorphism of Q, defined now by

$$\sigma(i) = j, \ \sigma(j) = k^{-1} = -k, \ \sigma(k) = i \ .$$

The two transformations differ by the cyclic transformation

$$\tau : i \mapsto j \mapsto k \mapsto i$$

which group-theorists call an outer automorphism, of order three, of the quaternion group Q: in these terms, $\lambda = \tau \circ \sigma$. The point is that the symmetries [17] of Q form a larger group; if we include anti-automorphisms among them, we get a very interesting group of order twenty-four, in which both transformations λ and σ might be understood as playing a distinguished role.

8 Perhaps it will be useful to mention that modern mathematical logic (cf. eg [6]) is very sophisticated, and is willing to study systems with 'truth-values' in quite general commutative groups, in a way entirely consistent with Chris Gregory's Ramusian [4] precepts; but to my knowledge, logic with values in **non**-commutative groups has been studied only in contexts motivated by higher mathematics (see eg [9]). This may explain, to some extent, the difficulty people have had, in finding a mathematical interpretation of Lévi-Strauss's ideas; but it seems clear to me that such an interpretation does exist, and that as far as I can see, it fits integrally with Lèvi-Strauss's earlier work on the subject.

I hope those who read this will not be offended if I close with a personal remark. When I first encountered Lévi-Strauss's formula, my reaction was bemusement and skepticism; I took the question seriously, in large part because I was concerned that it might represent an aspect of some kind of anthropological cargo-cult, based on a fetishization of mathematical formalism. I am an outsider to the field, and can make judgements of Lévi-Strauss's arguments only on the basis of internal consistency

(in so far as I am competent to understand them); but I have to say that I am now convinced that the man knows his business.

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